stichting
mathematisch
centrum

DEPARTMENT OF PURE MATHEMATICS ZW 64/76 MARCH

A.E. BROUWER & A. SCHRIJVER

A GROUP-DIVISIBLE DESIGN GD(4,1,2;n) EXISTS IFF $n \equiv 2 \pmod 6$, $n \neq 8$ (OR: THE PACKING OF COCKTAIL PARTY GRAPHS WITH K_4'S)

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 05B30, 05B40, (05C35)

A group-divisible design GD(4,1,2;n) exists iff $n \equiv 2 \pmod 6$, $n \neq 8$ (or: the packing of cocktail party graphs with K_4's)

by

A.E. Brouwer & A. Schrijver

ABSTRACT

In this paper it is shown that for $n \equiv 2 \pmod 6$, $n \neq 8$, the complete graph K_n can be partitioned into $n(n-2)/12$ copies of K_4 and a 1-factor (matching). It follows that the maximum cardinality of a binary constant-weight code with minimum distance $d = 6$ and words of weight 4 is $n(n-2)/12$ for these values of n. The methods used are explicit construction and recursive techniques, as developed by Hanani and Wilson.

KEY WORDS & PHRASES: *group-divisible design, scarce design, packing, constant-weight code.*

0. INTRODUCTION

Let I_n be the set $\{0,\dots,n-1\}$. For $n \geq k \geq t$ let $C(n,k,t)$ (resp. $D(n,k,t)$) be the smallest (resp. largest) integer b such that there exist b subsets $B_1,\dots,B_b$ of I_n, each of k elements, such that every t-element subset of I_n is contained in at least (resp. at most) one of them.

Many authors have determined the value of $C(n,k,t)$ or $D(n,k,t)$ for special values of n, k and t. In particular FORT & HEDLUND [1] have shown that

$$C(n,3,2) = \left\lceil \frac{n}{3} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil \qquad \text{for } n \geq 3.$$

In 1966 SCHÖNHEIM [8] showed that

$$D(n,3,2) = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1 & \text{for } n \equiv 5 \pmod 6, \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor & \text{otherwise.} \end{cases}$$

(This was done independently, but later, by GUY [2] in 1967 and SPENCER [9] in 1968. The solution of KIRKMAN [4] in 1847 is correct for $n \equiv 0,1,2,3 \pmod 6$, but false for $n \equiv 4$ or $5 \pmod 6$.)

MILLS [5] has shown that $C(n,4,2) = \left\lceil \frac{n}{4} \left\lceil \frac{n-1}{3} \right\rceil \right\rceil$ for $n \geq 4$ except for $n = 7,9,10,19$, while for $n = 7,9,10$ $C(n,4,2)$ is one more, and for $n = 19$ two more than the value given by this formula.

Mills also determined many values of $C(n,4,3)$ (see [6]).

In this paper we do part of the job of computing $D(n,4,2)$ by proving that

$$D(n,4,2) = \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor = \frac{1}{12} n(n-2) \qquad \text{for } n \equiv 2 \pmod 6,\ n \neq 8,$$

while

$$D(8,4,2) = 2.$$

In a subsequent paper we will show that

$$D(n,4,2) = \begin{cases} \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12}, \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor & \text{otherwise,} \end{cases}$$

with a few exceptions. (For sufficiently large n this result follows immediately from the theory developed by WILSON (see, e.g., [10]); the difficult part is to fill in the finite gap left.)

This problem can be described in several other ways. Determining $D(n,k,t)$ amounts to the same as finding an optimal binary constant weight code with word length n, constant weight k and minimum distance $2(k-t+1)$.

For $t = 2$, $C(n,k,2)$ (resp. $D(n,k,2)$) is the number of complete graphs K_k required in order to cover (the edges of) K_n (resp. which can be packed (edge-disjoint) in K_n).

Looking in particular at the case $k = 4$, $t = 2$, $n \equiv 2 \pmod 6$ we see that if we pack copies of K_4 into K_n, then each K_4 uses three edges incident with a given point p. Since K_n is regular with valency $n-1 \equiv 1 \pmod 6$, we see that a packing must necessarily leave unused at least one edge for each vertex.

We will show that except for $n = 8$ there exists a packing that uses all edges except for a one-factor, so that

$$D(n,4,2) = \frac{1}{6}\left(\binom{n}{2} - \tfrac{1}{2}n\right) = n(n-2)/12, \quad \text{for } n \equiv 2 \ (6), \quad n \neq 8.$$

(Using Hoffman's terminology this means that we have for these values of n a packing of a cocktail party graph on n points with K_4's.)

In this case (and more generally in case $t = 2$, n in a suitable congruence class mod $k(k-1)$) again different terminology is possible:

A *group-divisible design* $GD(K,\lambda,M;v)$ is a pair $(\mathcal{B},\mathcal{G})$ of collections of subsets of I_v (called *blocks* resp. *groups*) such that

(i) if $B \in \mathcal{B}$, then $|B| \in K$;

(ii) if $G \in \mathcal{G}$, then $|G| \in M$;

(iii) if $\{i,j\}$ is a 2-element subset of I_v, then either there is exactly one group $G \in \mathcal{G}$ such that $\{i,j\} \subset G$ or there are exactly λ blocks

$B \in \mathcal{B}$ such that $\{i,j\} \subset B$;

(iv) if $G_1, G_2 \in \mathcal{G}$, then $G_1 \cap G_2 = \phi$;

(v) $\cup\mathcal{G} = I_v$.

Instead of GD($\{k\},\lambda,\{m\};v$) we write GD($k,\lambda,m;v$).
A *transversal design* TD($k,\lambda;m$) is a group-divisible design GD($k,\lambda,m;km$) (cf. HANANI [3]).

From the description above it is immediately seen that the determination of D(n,4,2) for $n \equiv 2 \pmod 6$ requires the construction of GD(4,1,2;n). This is done in the remaining part of this paper, thus proving the 'if' part of:

THEOREM. *A* GD(4,1,2;n) *exists iff* $n \equiv 2 \pmod 6$, $n \neq 8$.

The 'only if' part is immediate since

(a) A GD(4,1,2;8) would be equivalent to a set of two mutually orthogonal Latin squares of side 2, which does not exist.

(b) If GD(4,1,2;n) exists, then
   (i) n is even, since I_n can be partitioned into groups of size 2, and
   (ii) $n \equiv 2 \pmod 3$, since the n-1 edges incident with a given point are covered by one K_2 (one edge) and a number of K_4's (three edges each).

   Therefore, $n \equiv 2 \pmod 6$.

1. THE TRUNCATED TRANSVERSAL DESIGN

THEOREM. *Let* $h \le t$, $2t \in T(5,1)$, $\{6h+2, 6t+2\} \subset GD(4,1,2)$. *Then* $6(4t+h)+2 \in$ $\in GD(4,1,2)$.

PROOF. Let T be a set with cardinality $|T| = 2t$, and let $\mathcal{T}$ be a transversal design T(5,1;2t) on the set $T \times I_5$ with groups $T \times \{i\}$, $i \in I_5$. Let $H \subset T$ be a subset of cardinality $|H| = 2h$. Then we have the group-divisible design GD($\{4,5\},1,\{2t,2h\}$) on $X = (T \times I_4) \cup (H \times \{4\})$, given by $\mathcal{T}_1 = \{B \in \mathcal{T},\ B \subset X\}$, with groups $T \times \{i\}$, $i \in I_4$ and $H \times \{4\}$.
Now let $Y = (X \times I_3) \cup Z$, where $Z = I_2$, and construct a GD(4,1,2) on Y by taking the groups and blocks of the GD(4,1,2) on the sets $(T \times \{i\} \times I_3) \cup Z$, $i \in I_4$, and $(H \times \{4\} \times I_3) \cup Z$, taking care that each of these GD(4,1,2)

contains Z as a group, and furthermore the blocks of a GD(4,1,3) on each of the sets $B \times I_3$ where $B \in T_1$, taking care that each of these GD(4,1,3) contains $\{b\} \times I_3$ as a group for each $b \in B$. (Here we use the fact that $\{12,15\} \subset$ GD(4,1,3)). □

This theorem reduces the job of finding all GD(4,1,2) to a finite one. Let t be even, then $2t \in T(5,1)$ (see, e.g., [7]). Now each number r different from 1 can be written as $r = 4t + h$, t even, $h \in \{0,2,3,4,5,6,7,9\}$. If we assume that all designs GD(4,1,2;6s+2) have been constructed for $1 < s < r$, then by the theorem we find a GD(4,1,2;6r+2) provided that $t \geq h$. This means that we have to find GD(4,1,2;6s+2) for $s \in \{0,2,3,4,5,6,7,9,11,12,13,14,15,17,21,22,23,25,31,33,41\}$.

2. MULTIPLYING BY FOUR

THEOREM. *If* $v \in$ GD(4,1,2) *and* $v \neq 2$, *then* $4v \in$ GD(4,1,2).

PROOF. Let $|X| = v$ and construct a transversal design on $X \times I_4$ with groups $X \times \{i\}$, $i \in I_4$ (such a design exists since $v \notin \{2,6\}$); add to the blocks of this design the blocks and groups of group-divisible designs GD(4,1,2) on each of the sets $X \times \{i\}$ and we get a GD(4,1,2) on $X \times I_4$. □

If we write $v = 6r + 2$, then $4v = 6(4r+1) + 2$. Consequently, from the list given at the end of Section 1, we may drop all numbers congruent to 1 mod 4 (except 5). Therefore, it remains to find GD(4,1,2;6s+2) for $s \in \{0,2,3,4,5,6,7,11,12,14,15,22,23,31\}$.

3. THE CASE v = 2p

THEOREM. *Let* p *be a prime,* $p \equiv 1 \pmod 6$. *Then* $2p \in$ GD(4,1,2).

PROOF. Let $X = Z_p \times Z_2$ and let x be a primitive root mod p. Let $u = (p-1)/6$. Now consider the blocks

$$\{(i,1+k),(x^j+i,k),(x^{j+2u}+i,k),(x^{j+4u}+i,k)\} \quad (i \in Z_p,\ 0 \leq j < u,\ k \in Z_2),$$

and the groups

$$\{(i,0),(i,1)\} \qquad (i \in Z_p).$$

It is not difficult to check that this system gives a GD(4,1,2;2p). □

This disposes of all the even values of s:

s = 0 is the trivial case v = 2 (one group and no blocks),
s = 2 corresponds to v = 14 = 2.7,
s = 4 corresponds to v = 26 = 2.13,
s = 6 corresponds to v = 38 = 2.19,
s = 12 corresponds to v = 74 = 2.37,
s = 14 corresponds to v = 86 = 2.43, and
s = 22 corresponds to v = 134 = 2.67.

We are left with the case $s \in \{3,5,7,11,15,23,31\}$.

4. THE REMAINING SEVEN CASES

(i) The case s = 3, v = 20.
Let $X = I_4 \times Z_5$ and take the groups

$$\{(0,i),(3,i)\} \text{ and } \{(1,i),(2,i)\} \qquad (i \in Z_5),$$

and the blocks

$$\{(0,i),(0,i+1),(1,i+2),(1,i+4)\}$$
$$\{(0,i),(0,i+2),(2,i+3),(2,i+4)\}$$
$$\{(0,i),(1,i),(3,i+1),(3,i+4)\}$$
$$\{(0,i),(2,i),(3,i+2),(3,i+3)\}$$
$$\{(1,i+2),(1,i+3),(2,i),(3,i)\}$$
$$\{(1,i),(2,i+1),(2,i+4),(3,i)\} \qquad (i \in Z_5).$$

(ii) The case s = 5, v = 32.
Let $X = I_2 \times Z_{16}$ and take the groups

$\{(j,i),(j,i+8)\}$ $(i \in Z_{16},\ j \in I_2)$,

and the blocks

$\{(0,i),(0,i+6),(1,i),(1,i+2)\}$
$\{(0,i),(0,i+4),(0,i+11),(1,i+15)\}$
$\{(0,i),(0,i+1),(0,i+3),(1,i+6)\}$
$\{(0,i+8),(1,i),(1,i+1),(1,i+5)\}$
$\{(0,i+2),(1,i),(1,i+3),(1,i+9)\}$ $(i \in Z_{16})$.

(iii) The case $s = 7$, $v = 44$.
Let $X = I_2 \times Z_{22}$ and take the groups

$\{(j,i),(j,i+11)\}$ $(i \in Z_{22},\ j \in I_2)$,

and the blocks

$\{(0,i),(0,i+5),(1,i)(1,i+15)\}$
$\{(0,i),(0,i+8),(0,i+9),(1,i+6)\}$
$\{(0,i),(0,i+2),(0,i+6),(1,i+9)\}$
$\{(0,i),(0,i+7),(0,i+10),(1,i+11)\}$
$\{(0,i),(1,i+16),(1,i+18),(1,i+21)\}$
$\{(0,i),(1,i+2),(1,i+8),(1,i+12)\}$
$\{(0,i),(1,i+5),(1,i+13),(1,i+14)\}$ $(i \in Z_{22})$.

(iv) The case $s = 11$, $v = 68$.
Let $X = (Z_2 \times Z_3 \times Z_3 \times Z_3) \cup I_{14}$. We will construct a GD(4,1,2) on X by first taking a GD(4,1,2) on I_{14} and then covering $X \backslash I_{14}$ with a 1-factor (matching), 14 Δ-factors (partitions into triples) and 108 4-tuples. The 14 Δ-factors can then be completed to 14.18 4-tuples by adjoining one point of I_{14} to each of the triples in a Δ-factor.

1-factor : $\{(0,0,0,0),(1,0,0,0)\}$ mod $(-,3,3,3)$

Δ-factors: 1. $\{(0,0,0,0),(0,1,0,1),(0,2,0,2)\}$ mod $(2,3,3,3)$

(this gives only 18 triples, since each triple occurs thrice and we retain only one of each three identical triples).

2. $\{(0,0,0,0),(0,1,1,0),(0,2,2,0)\}$ mod $(2,3,3,3)$
3. $\{(0,0,0,0),(0,1,1,1),(0,2,2,2)\}$ mod $(2,3,3,3)$

4. $\{(0,0,0,0),(0,1,2,0),(0,2,1,0)\}$ mod $(2,3,3,3)$

5. $\{(0,0,0,0),(0,1,2,2),(0,2,1,1)\}$ mod $(2,3,3,3)$

6-8. $[\{(0,0,0,0),(1,0,1,0),(1,0,2,2)\}$ mod $(2,3,-,3)]$ mod $(-,-,3,-)$

9-11. $[\{(0,i,0,i),(1,i,0,i+1),(1,i+1,1,i)\}$ $(i=0,1,2)$ mod $(2,-,3,-)]$ mod $(-,3,-,-)$

12-14. $[\{(0,i,i,0),(1,i,i+1,2),(1,i+1,i,0)\}$ $(i=0,1,2)$ mod $(2,-,-,3)]$ mod $(-,3,-,-)$

4-tuples: $\{(0,0,0,0),(0,0,0,1),(0,1,0,0),(1,2,1,1)\}$ mod $(2,3,3,3)$
$\{(0,0,0,0),(0,0,1,1),(1,1,0,2),(1,1,1,2)\}$ mod $(2,3,3,3)$.

(v) The case $s = 15$, $v = 92$.
Let $X = I_4 \times I_{23}$. A GD(4,1,2) on X will be constructed with help of a GD(4,1,{2,5};23) with exactly one block of size 5 and a pair of orthogonal Latin squares of order 23 with orthogonal subsquares of order 5. Hence we first present both these systems.

(v.i) A GD(4,1,{2,5};23) with one block of size 5.
Let $Y = (Z_2 \times Z_3 \times Z_3) \cup I_5$. We will construct the GD(4,1,{2,5}) on Y by taking I_5 as a group and covering $Z_2 \times Z_3 \times Z_3$ with a 1-factor, 5 Δ-factors and 9 4-tuples, as follows:

1-factor: $\{(0,0,0),(1,0,0)\}$ mod $(-,3,3)$

Δ-factors:

1. $\{(0,0,0),(0,1,0),(0,2,0)\}$ mod $(2,-,3)$

2. $\begin{bmatrix}\{(0,0,2),(0,1,1),(0,2,0)\} & \text{mod } (-,3,-) \\ \{(1,0,0),(1,0,1),(1,0,2)\} & \text{mod } (-,3,-)\end{bmatrix}$

3-5. $[\{(0,0,0),(1,0,1),(1,1,2)\}$ mod $(2,3,-)]$ mod $(-,-,3)$

4-tuples: $\{(0,0,0),(0,0,1),(1,1,1),(1,2,0)\}$ mod $(-,3,3)$.

(v.ii) A T(4,1;23) with subdesign T(4,1;5).
This construction is a special case of the construction described by BOSE, PARKER & SHRIKHANDE. Take the affine plane AG(2,5) and delete two points; this gives a pairwise balanced design (cf. HANANI [3]) $\mathcal{B}$ = PBD({3,4,5},1;23) with one block of size 3 on a set I_{23}. Now construct a T(4,1;23) as follows:

for each block $B \in \mathcal{B}$ construct a $T(4,1;4 \cdot |B|)$ on $B \times I_4$ with groups $B \times \{i\}$, $i \in I_4$, taking care that if $|B| > 3$ then the sets $\{b\} \times I_4$, $b \in B$, are blocks of the $T(4,1;4 \cdot |B|)$. Taking all blocks of the thus constructed transversal designs except for the three blocks of the type $\{b\} \times I_4$ with $b \in B_0$, the unique block of size 3, we get a $T(4,1;23)$. It contains subdesigns $T(4,1;5)$ since $\mathcal{B}$ contains blocks of size 5 disjoint from the block of size 3.

(v.iii) A $GD(4,1,2;92)$.
Let $\mathcal{T}$ be a transversal design $T(4,1;23)$ on $X = I_{23} \times I_4$ with a subdesign $\mathcal{T}_0 = T(4,1;5)$ on $Y = I_5 \times I_4$. Let $\mathcal{B}_i$ be a $GD(4,1,\{2,5\};23)$ on $I_{23} \times \{i\}$ with one group of size 5, say $I_5 \times \{i\}$, $(i \in I_4)$. Finally let $\mathcal{D}$ be a $GD(4,1,2;20)$ on Y. Taking the blocks of $\mathcal{T}\backslash\mathcal{T}_0$ and those of $\mathcal{B}_i$ $(i \in I_4)$ and $\mathcal{D}$, and the groups of size 2 of $\mathcal{B}_i$ $(i \in I_4)$ and $\mathcal{D}$, we get a $GD(4,1,2)$ on X.

(vi) The cases $s = 23$ $(v = 140)$ and $s = 31$ $(v = 188)$.
Set for $v = 140, 188$: $v = 48m - 4$ where $m = 3,4$.
Take a resolvable design $RB(4,1;12m+4)$, and form a partial completion by adjoining $4m-6$ points p_i $(i = 1,\dots,4m-6)$, where the point p_i is adjoined to each of the blocks of the i-th parallel class. (This is allowed since there are $4m+1$ parallel classes.) In this way we get a $B(\{4,5,4m-6\},1;16m-2)$ on a set I_n where $n = 16m-2$. Since we did not use all parallel classes of the original design, we can pick the blocks of one parallel class together with the block of size $4m-6$ and call them groups. This gives us a $GD(\{4,5\},1,\{4,4m-6\};n)$. Now let $X = (I_n \times I_3) \cup I_2$ and form a $GD(4,1,3;12)$ resp. $GD(4,1,3;15)$ on $B \times I_3$ for each block B of this design, taking care that each time $\{b\} \times I_3$ becomes a group for each $b \in B$; also form $GD(4,1,2;14)$ resp. $GD(4,1,2;12m-16)$ on $(G \times I_3) \cup I_2$ for each group G of this design, taking care that each time I_2 becomes a group. If we take all the blocks thus obtained, and all groups of size two (where of course I_2 is taken only once) we get a $GD(4,1,2;48m-4)$.

This settles all cases.

REFERENCES

[1] FORT Jr., M.K. & G.A. HEDLUND, *Minimal coverings of pairs by triples*, Pacif. J. Math. 8 (1958) 709-719.

[2] GUY, R.K., *A problem of Zarankiewicz*, Research paper no 12, Univ. of Calgary (1967).

[3] HANANI, H., *Balanced incomplete block designs and related designs*, Discr. Math. 11 (1975) 255-369.

[4] KIRKMAN, T.P., *On a problem in combinations*, Cambridge and Dublin Mathematical Journal 2 (1847) 191-204.

[5] MILLS, W.H., *On the covering of pairs by quadruples I, II*, J. Combinatorial Theory (A) 13 (1972) 55-78 and 15 (1973) 138-166.

[6] MILLS, W.H., *On the covering of triples by quadruples*, Proc. 5th S.E. Conf. on Combinatorics, Graph Theory and Computing (1974) 563-581.

[7] MILLS, W.H., *Three mutually orthogonal Latin squares*, J. Combinatorial Theory (A) 13 (1972) 79-82.

[8] SCHÖNHEIM, J., *On maximal systems of k-tuples*, Studia Scientiarum Mathematicarum Hungarica 1 (1966) 363-368.

[9] SPENCER, J., *Maximal consistent families of triples*, J. Combinatorial Theory 5 (1968) 1-8.

[10] WILSON, R.M., *The construction of group divisible designs and partial planes having the maximum number of lines of a given size*, Proc. Chapel Hill, 1970, pp. 488-497.